ON THE GROUP OF REAL LINEAR TRANSFORMATIONS WHOSE INVARIANT IS A REAL QUADRATIC FORM.

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In what follows G will denote the group of real linear homogeneous transformations of determinant $+1$ whose invariant is the real quadratic form of non-zero determinant,

$$\mathfrak{F} = (\Omega \between x_1, x_2, \ldots x_n)^2,$$

where Ω is a real symmetric matrix.*

Since the quadratic form $\mathfrak{F}$ is real, the roots of its characteristic equation are all real. In the *Proceedings of the London Mathematical Society*, Vol. XXVI., page 376, I have shown that for $n = 4$ the group of all proper linear homogeneous transformations, real and imaginary, whose invariant is the real quadratic form $\mathfrak{F}$, provided the roots of the characteristic equation of $\mathfrak{F}$ are not all of the same sign, contains a *real* transformation that cannot be generated by the repetition of any infinitesimal transformation of this group; and that therefore, *a fortiori*, cannot be generated by the repetition of an infinitesimal transformation of group G. It follows that for $n > 4$ not every transformation of group G can be generated by the repetition of an infinitesimal transformation of group G if the roots of the characteristic equation of $\mathfrak{F}$ are not all of the same sign.

On the other hand, if the roots of the characteristic equation of $\mathfrak{F}$ are all positive or all negative, every transformation of group G can be generated by the repetition of an infinitesimal transformation of this group.†

* I employ the notation of Cayley's "Memoir on the Automorphic Linear Transformation of a Bipartite Quadric Function," *Philosophical Transactions*, 1858, with this exception: the transverse of a matrix or linear transformation ϕ will be denoted by $\breve{\phi}$.

† If the roots of the characteristic equation of $\mathfrak{F}$ are all of the same sign, group G is isomorphic with the group of real proper orthogonal substitutions; and every

I shall therefore assume that $n \geqq 4$, that the roots of the characteristic equation of $\mathfrak{F}$ are not all of the same sign; and I shall show that a transformation of group G can be generated by the repetition of an infinitesimal transformation of group G if it is an even power of a transformation of this group.

If the matrix or linear homogeneous transformation ϕ transforms $\mathfrak{F}$ automorphically, it satisfies the matrical equation

$$\breve{\phi}\, \Omega\, \phi = \Omega, \tag{1}$$

in whch $\breve{\phi}$ denotes the transverse or conjugate of ϕ. Conversely, if this equation is satisfied, ϕ transforms $\mathfrak{F}$ automorphically. The determinant of any transformation satisfying this equation is equal to either $+1$ or -1. By definition the totality of real proper solutions of equation (1) constitutes group G.

If ϕ is any real solution of equation (1), we may put

$$\phi = \phi_0\, \phi_1 = \phi_1\, \phi_0,$$

where ϕ_0, ϕ_1, are polynomials in ϕ, are both real, and are both solutions of equation (1), that is

$$\breve{\phi}_0\, \Omega\, \phi_0 = \Omega, \qquad \breve{\phi}_1\, \Omega\, \phi_1 = \Omega.$$

Moreover -1 is not a root of the characteristic equation of ϕ_1, therefore the determinant of ϕ_1 is equal to $+1$;* and consequently ϕ_1 is a transformation of group G. Finally,

$$\phi_0^2 = 1;$$

and therefore

$$\phi^2 = \phi_0^2\, \phi_1^2 = \phi_1^2. \tag{2}$$

That is, the second power of any real solution of equation (1) is the second power of a transformation of group G.†

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transformation of this group can be generated by the repetition of a real infinitesimal orthogonal transformation.

* The roots, other than ± 1, of the characteristic equation of any solution of equation (1) occur in pairs, the product of two of the same pair being unity. The determinant of a linear transformation is equal to the product of the roots of its characteristic equation.

† If -1 is a root of multiplicity m of the characteristic equation of ϕ, and if the roots of this equation other than -1 are g_1, g_2, etc. of multiplicity, respectively, p_1, p_2, etc., then

$$\phi_0 = 1 - 2\,\Phi,$$

where

$$\Phi = \frac{[(\phi + 1)^m - (g_1 + 1)^m]^{p_1}}{[-(g_1 + 1)^m]^{p_1}} \cdot \frac{[(\phi + 1)^m - (g_2 + 1)^m]^{p_2}}{[-(g_2 + 1)^m]^{p_2}} \cdots$$

Let now e^{χ} denote the infinite series

$$1 + \chi + \frac{1}{2\,!}\chi^2 + \frac{1}{3\,!}\chi^3 + \text{etc.},$$

convergent for any finite matrix. We have

$$(e^{\chi})^{-1} = e^{-\chi},$$

$$\breve{(e^{\chi})} = e^{\breve{\chi}},$$

and for any integer m,

$$(e^{\chi})^m = e^{m\chi};$$

moreover, if χ and χ' are commutative,

$$e^{\chi + \chi'} = e^{\chi}\, e^{\chi'} = e^{\chi'}\, e^{\chi}.$$

Corresponding to any finite matrix, ϕ, of non-zero determinant can be found a polynomial χ in ϕ such that

$$\phi = e^{\chi}.$$

The infinite series

$$1 - \frac{1}{2\,!}\chi^2 + \frac{1}{4\,!}\chi^4 - \text{etc.}$$

and

$$\chi - \frac{1}{3\,!}\chi^3 + \frac{1}{5\,!}\chi^5 - \text{etc.}$$

are also convergent for any finite matrix, and are equal respectively to

$$\frac{1}{2}(e^{\chi\sqrt{-1}} + e^{-\chi\sqrt{-1}}), \quad \frac{1}{2\sqrt{-1}}(e^{\chi\sqrt{-1}} - e^{-\chi\sqrt{-1}}).$$

Therefore, if χ and $e^{\chi\sqrt{-1}}$ are both real, the second power of the latter is equal to the identical transformation. For if χ and $e^{\chi\sqrt{-1}}$ are real,

$$e^{\chi\sqrt{-1}} - e^{-\chi\sqrt{-1}} = 0;$$

that is,

$$(e^{\chi\sqrt{-1}})^2 = 1.$$

Since the determinant of ϕ_1 is not zero, by what precedes, a polynomial χ in ϕ_1 can be found such that

$$\phi_1 = e^{\chi};$$

and since -1 is not a root of the characteristic equation of ϕ_1, χ may be so chosen that, if

$$\vartheta = \chi\,\Omega^{-1},$$

we shall have

$$\breve{\vartheta} = -\vartheta,$$

that is, ϑ is skew symmetric. If now

$$\vartheta = \theta + \eta\sqrt{-1},$$

where θ and η are real, both θ and η are skew symmetric, that is,

$$\breve{\theta} = -\theta, \quad \breve{\eta} = -\eta. \tag{3}$$

Since Ω is real, $\theta\,\Omega$ and $\eta\,\Omega\sqrt{-1}$ are the real and imaginary parts respectively of $\vartheta\,\Omega = \chi$. And since the latter is a polynomial in the real matrix ϕ_1, its real and imaginary parts, $\theta\,\Omega$ and $\eta\,\Omega\sqrt{-1}$, are polynomials in ϕ_1, and are therefore commutative. Consequently, by virtue of a theorem given above,

$$\begin{aligned}\phi_1 &= e^{\chi} \\ &= e^{\vartheta\,\Omega} \\ &= e^{\theta\,\Omega + \eta\,\Omega\sqrt{-1}} = e^{\theta\,\Omega}\,e^{\eta\,\Omega\sqrt{-1}}.\end{aligned}$$

Since ϕ_1 is real, and since $\theta\,\Omega$ and therefore $e^{\theta\,\Omega}$ is real, it follows that $e^{\eta\,\Omega\sqrt{-1}}$ is real. Therefore, by what precedes, since $\eta\,\Omega$ is real,

$$(e^{\eta\,\Omega\sqrt{-1}})^2 = 1.$$

Whence we have

$$\phi_1^{\,2} = (e^{\theta\,\Omega})^2\,(e^{\eta\,\Omega\sqrt{-1}})^2 = (e^{\theta\,\Omega})^2 = e^{2\,\theta\,\Omega};$$

and therefore by (2)

$$\phi^2 = \phi_1^{\,2} = e^{2\,\theta\,\Omega}.$$

If now we put

$$\psi = e^{\frac{2}{m}\theta\,\Omega},$$

where m is any positive integer, ψ is real, and

$$\psi^m = (e^{\frac{2}{m}\theta\,\Omega})^m = e^{2\,\theta\,\Omega} = \phi^2.$$

Moreover, since $\breve{\Omega} = \Omega$, and since, by (3), $\breve{\theta} = -\theta$, we have

$$\breve{\psi} = e^{\frac{2}{m}\breve{\overline{\theta\,\Omega}}} = e^{\frac{2}{m}\breve{\Omega}\breve{\theta}} = e^{-\frac{2}{m}\Omega\,\theta};$$

and therefore

$$\begin{aligned}\breve{\psi}\,\Omega\,\psi &= e^{-\frac{2}{m}\Omega\,\theta}\,\Omega\,e^{\frac{2}{m}\theta\,\Omega} \\ &= \Omega\,e^{-\frac{2}{m}\theta\,\Omega}\,e^{\frac{2}{m}\theta\,\Omega} = \Omega.*\end{aligned}$$

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* For any positive integer p,

$$(\Omega\,\theta)^p\,\Omega = \Omega\,(\theta\,\Omega)^p.$$

Therefore

$$e^{-\frac{2}{m}\Omega\,\theta}\,\Omega = \Omega\,e^{-\frac{2}{m}\theta\,\Omega}.$$

Finally, we may show in precisely the same way that $e^{\frac{1}{m}\theta\Omega}$ is a real solution of equation (1); and therefore, since $(e^{\frac{1}{m}\theta\Omega})^2 = e^{\frac{2}{m}\theta\Omega}$, it follows that ψ is the second power of a solution of equation (1), and is thus of determinant $+1$. Wherefore, ψ is a transformation of group G.*

By taking m sufficiently great, $\frac{2}{m}\theta\Omega$ may be made as nearly as we please equal to zero, and therefore $\psi = e^{\frac{2}{m}\theta\Omega}$ may be made as nearly as we please equal to the identical transformation. Wherefore, $\phi^2 = \phi_1{}^2$ can be generated by the repetition of an infinitesimal transformation of group G.

The converse is of course also true; that is to say, every transformation generated by the repetition of an infinitesimal transformation of group G is the pth power (for any positive integer p), and therefore the second power, of a transformation of this group. This may be shown as follows. The transformation ϕ, if infinitesimal, may be put equal to $1 + \delta t \,.\, \chi$, where δt is infinitesimal. If ϕ is a transformation of group G, δt and χ are both real, and we have

$$(1 + \delta t \,.\, \breve{\chi})\,\Omega\,(1 + \delta t \,.\, \chi) = \breve{\phi}\,\Omega\,\phi = \Omega\,.$$

That is, neglecting terms involving the second power of δt,

$$\breve{\chi}\,\Omega + \Omega\,\chi = 0\,.$$

If we put $\theta = \chi\,\Omega^{-1}$, θ is real, and the preceding equation becomes

$$\Omega\,\breve{\theta}\,\Omega + \Omega\,\theta\,\Omega = 0\,;$$

that is,

$$\breve{\theta} = -\theta\,.$$

The transformation resulting from m repetitions of the infinitesimal transformation

$$\phi = 1 + \delta t \,.\, \chi = 1 + \delta t \,.\, \theta\,\Omega$$

is the transformation $(1 + \delta t \,.\, \theta\,\Omega)^m$, which, if m is infinite, is equal to $e^{m\,\delta t\,.\,\theta\Omega}$, or $e^{t\,\theta\,\Omega}$ if we put $t = m\,\delta t$. The repetition of an infinitesimal transformation of group G results in a transformation of this group; and therefore, as we have in fact already seen, $e^{t\,\theta\,\Omega}$ for any real quantity t is a

* The matrix $e^{\frac{1}{m}\theta\Omega}$ is also a transformation of group G. The totality of transformations $e^{\frac{1}{m}\theta\Omega}$ for all possible real values of m constitutes a continuous one term sub-group which contains the identical transformation.

transformation of group G. But then $e^{\frac{t}{p}\theta\Omega}$, for any positive integer p, is a transformation of group G; and since $(e^{\frac{t}{p}\theta\Omega})^p = e^{t\theta\Omega}$, any transformation generated by the repetition of an infinitesimal transformation of group G is the pth power, for any integer p, of a transformation of this group.

It may be shown that any transformation of group G that cannot be generated by the repetition of an infinitesimal transformation of group G is the $(2p+1)$th power of a transformation of this group for any integer p.

For any transformation of group G that can be generated by the repetition of an infinitesimal transformation of group G, the numbers belonging to each negative root of the characteristic equation are all even.*

Postscript.

Let $\mathfrak{G}$ denote the group of all linear homogeneous transformations, real and imaginary, of determinant $+1$ whose invariant is the real quadratic form of non-zero determinant,

$$\mathfrak{F} = (\Omega \between x_1, x_2, \ldots x^n)^2 ;$$

and, as above, let G denote the sub-group of real transformations of group $\mathfrak{G}$.

A transformation of group $\mathfrak{G}$ can be generated by the repetition of an infinitesimal transformation of this group if -1 is not a root of the characteristic equation of the transformation, or if -1 is a root of the characteristic equation, provided the numbers belonging to -1 are all even.†

As stated above, if a transformation of group G can be generated by the repetition of an infinitesimal transformation of this group, the numbers belonging to each negative root of the characteristic equation of the transformation are all even. These conditions, though necessary, are not always sufficient. Thus, for $n = 2$, the form $\mathfrak{F}$ is transformed automorphically if

$$x'_1 = -x_1, \quad x'_2 = -x_2 ;$$

* For definition of the numbers belonging to a root of the characteristic equation of a transformation, see These Preceedings, Vol. XXXI. p. 336.

† *Proceedings of the London Mathematical Society*, Vol. XXVI. p. 374.

and this transformation of group G, if the roots of the characteristic equation of $\mathfrak{F}$ are not both of the same sign, is not the second power of any transformation of group G, and therefore cannot be generated by the repetition of an infinitesimal transformation of this group. Since the numbers belonging to -1 are all even, the transformation defined by the above equations can be generated by the repetition of an infinitesimal transformation of group $\mathfrak{G}$.

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